

# One-point statistics of the induced electric field in quasinormal magnetofluid turbulence

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We study one-point statistical properties of the induced turbulent electric field for a magnetohydrodynamic (MHD) plasma under the quasinormal approximation. Assuming exact Gaussianity for both the velocity field and the magnetic field, and different degrees of correlations between their Cartesian components, we derive the probability distribution function (PDF) for the Cartesian components of the electric field  $e_i$ . We show that the PDF reduces in some canonical cases to an exponential function of the form  $\exp(-|e_i|)$ . To study deviations from these results in the more realistic case in which the velocity and magnetic fields are not exactly normal but quasinormal instead, we perform three-dimensional numerical simulations of the MHD equations at moderate Reynolds numbers. For turbulent relaxation from an initial condition, we find that the analytical results give a very good first-order approximation to the computed PDF.

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## I. INTRODUCTION

The induced electric field plays a key role in the slow evolution of a plasma. Such an evolution is usually described by the magnetohydrodynamic (MHD) equations. In these equations, the electric field  $\mathbf{E}$  is eliminated in terms of the magnetic field  $\mathbf{B}$  by means of a simplified Ohm's law,

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \eta \mathbf{j}, \quad (1)$$

where  $\eta$  is the resistivity,  $\mathbf{j} = \nabla \times \mathbf{B}$  is the electric current, and  $\mathbf{V}$  is the velocity field. The first term in the right-hand side (rhs) of Eq. (1) is referred to as the induced electric field. Ohm's law allows us to write the MHD induction equation

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad (2)$$

which states that the time evolution of the magnetic field is governed by the curl of the induced electric field.

Here we study the statistical properties of the induced electric field in fully developed MHD turbulence. In this scenario, both the velocity and the magnetic field are usually modeled as homogeneous, isotropic random variables. Nonetheless, in more realistic situations the dynamical fields need to be decomposed as

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{v}, \quad (3)$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad (4)$$

where  $\mathbf{V}_0 = \langle \mathbf{V} \rangle$ ,  $\mathbf{B}_0 = \langle \mathbf{B} \rangle$ , and the  $\langle \rangle$  operator denotes an ensemble average. Note that, by definition

$$\langle \mathbf{v} \rangle = 0 = \langle \mathbf{b} \rangle. \quad (5)$$

It is important to note that the statistical behavior of  $\mathbf{v}$  and  $\mathbf{b}$  exerts an influence on the evolution of the mean fields. For instance, taking the ensemble average of Eq. (2) we obtain

$$\partial_t \mathbf{B}_0 = \nabla \times (\mathbf{V}_0 \times \mathbf{B}_0) + \nabla \times \langle \mathbf{v} \times \mathbf{b} \rangle + \eta \nabla^2 \mathbf{B}_0. \quad (6)$$

The second term in the rhs of Eq. (6) reflects the action of the mean induced electric field of the fluctuations on the mean

magnetic field. This term is responsible for the generation of large scale magnetic fields from a turbulent velocity field, a phenomenon known as the "dynamo effect" (see, for instance, Ref. [1]). It is this fluctuating component of the electric field

$$\mathbf{e} = -\mathbf{v} \times \mathbf{b} \quad (7)$$

that we are interested in in this paper. Not only its mean value drives the evolution of the mean magnetic field, but  $\mathbf{e}$  itself drives the evolution of the fluctuating magnetic field, as can be seen by subtracting Eq. (6) from Eq. (2). Thus the statistics of  $\mathbf{e}$  may be expected to drive the statistics of  $\partial_t \mathbf{b}$  in a way similar to that by which the statistics of pressure and advection drive the statistics of  $\partial_t \mathbf{v}$  in hydrodynamics [2]. Beyond the scope of MHD, the turbulent induced electric field may also have an (as yet unexplored) important role in charged particle acceleration, transport, and diffusion.

Despite its relevance, little effort has been devoted to study the statistical properties of the turbulent induced electric field. To our knowledge, the only astrophysical observational paper reporting statistics of turbulent electric fields is Ref. [3] for the solar wind. This context, the solar wind, provides perhaps the best opportunity for direct observational study of the active plasma turbulence, including the turbulent induced electric field. A Kolmogorov-like power spectrum has been reported for the components of  $\mathbf{e}$ , but no information on the probability distribution function (PDF) of  $\mathbf{e}$  is given. Numerical simulations of the particle acceleration in turbulent reconnection [4] show in fact histograms for  $\mathbf{e}$ , which are more peaked than a Gaussian, but poorly resolved.

In principle, all the one-point statistical properties of  $\mathbf{e}$  can be derived from its PDF. The PDF for  $\mathbf{e}$ , in turn, may be computed in terms of the joint PDFs for  $\mathbf{v}$  and  $\mathbf{b}$ . As a natural starting point, we assume that both  $\mathbf{v}$  and  $\mathbf{b}$  are Gaussian random variables. It has been long known that the velocity field in the fluid turbulence may be regarded as a quasinormal, random (vector) variable. Early experiments (see, for example, Ref. [5]) demonstrated that the PDFs for the components of the velocity field are very close to Gaussian distributions, and that the kurtosis  $[K(x) = \langle x^4 \rangle / \langle x^2 \rangle^2]$ , where

$\langle x \rangle = 0$ ] of these components is accordingly very close to the Gaussian value of 3. This result is also valid in MHD, where both  $\mathbf{v}$  and  $\mathbf{b}$  are approximately Gaussian (see, for instance, [6]). There is also substantial direct observational evidence that the magnetic field fluctuations in the solar wind are normal to a good approximation [7,8].

It is worth mentioning that many closure theories for turbulence, such as the eddy damped quasilinear Markovian (EDQNM) model (see, for instance, [6] for a review) rely on the quasilinear approximation [9,10], which consists of the assumption that fourth-order moments of the fields  $\mathbf{v}$  and  $\mathbf{b}$  are exactly Gaussian, which closes the infinite hierarchy of statistical moments. Here, instead, we simply assume exact Gaussianity of these fields for the sole purpose of obtaining an approximate PDF for the electric field. An important remark needs to be made at this point. Closure theories of turbulence deal with two-point correlations, and thus are directly related to two-point PDFs. The statistics in this case turn out to be clearly non-Gaussian when the separation between the two points is in the dissipation range. However, as the separation between the two points tends to the integral scale, the points become uncorrelated and the one-point quasi-Gaussian statistics are recovered [6,11]. Remarkably, even though a considerable amount of research has been done in one-point and two-point statistics of the velocity and magnetic fields in magnetofluid turbulence, the statistics of the turbulent induced electric field have almost been ignored. We hope that this paper starts filling this gap.

The manuscript is organized as follows. In Sec. II we seek for analytical expressions for the PDF of the components of  $\mathbf{e}$  under the assumption of different degrees of correlation between the components of  $\mathbf{v}$  and  $\mathbf{b}$ . A preliminary numerical examination of these results is performed in Sec. III. Finally, we summarize the results and present our conclusions in Sec. IV.

## II. ANALYTICAL MODEL FOR THE PDF OF THE INDUCED ELECTRIC FIELD

We want to obtain an expression for the PDF of any component of  $\mathbf{e}$ , say

$$e_l = v_i b_j - v_j b_i, \quad (8)$$

where the cyclic indices  $\{l, i, j\}$  all take different values between 1 and 3, according to  $\mathbf{e} = -\mathbf{v} \times \mathbf{b}$ . For simplicity in the notation, we will consider a random variable  $z$  of the form

$$z \equiv x_1 x_2 - x_3 x_4, \quad (9)$$

which simply reflects the change in variables:  $\{e_l, v_i, b_j, v_j, b_i\} \rightarrow \{z, x_1, x_2, x_3, x_4\}$ . An important simplification comes from the hypothesis of variance isotropy [12] (that we will use throughout the paper, unless explicitly stated). Variance isotropy, that is,  $\sigma_{v_i} = \sigma_{v_j}$  and  $\sigma_{b_i} = \sigma_{b_j}$  for all  $i, j = 1, 2, 3$  implies  $\sigma_{v_i} \sigma_{b_j} = \sigma_{v_j} \sigma_{b_i}$ , or in terms of Eq. (9),

$$\sigma_1 \sigma_2 = \sigma_3 \sigma_4, \quad (10)$$

where  $\sigma_x = \sigma(x)$  stands for the standard deviation of  $x$  and  $\sigma_i \equiv \sigma(x_i)$ .

### A. Uncorrelated Gaussians

Calculations are much simplified under the hypothesis of statistical independence of the variables  $x_1, x_2, x_3, x_4$ . Even though many astrophysical and laboratory plasmas have an appreciable amount of correlation between  $\mathbf{v}$  and  $\mathbf{b}$ , the working assumption of uncorrelated fields is often used for theoretical modeling and numerical exploration of MHD turbulence due to its relative simplicity. We use it here to introduce our calculations in the simplest possible context.

A first point to note is that if  $x_1$  and  $x_2$  are independent, then

$$\langle (x_1 x_2)^n \rangle = \langle x_1^n \rangle \langle x_2^n \rangle. \quad (11)$$

This expression, in turn, simplifies the calculation of the moments of combinations of the various  $\{x_i\}$ . In particular,

$$\sigma \langle x_1 x_2 \rangle = \sigma_1 \sigma_2, \quad (12)$$

$$K(x_1 x_2) = K_1 K_2 = 9, \quad (13)$$

$$\sigma(x_1 x_2 - x_3 x_4) = \sqrt{2} \sigma_1 \sigma_2, \quad (14)$$

$$K(x_1 x_2 - x_3 x_4) = \frac{1}{2} K_1 K_2 + \frac{3}{2} = 6, \quad (15)$$

where  $K_x = K(x)$  is the kurtosis of  $x$  and  $K_i \equiv K(x_i)$ . The numerical values given for the kurtosis only hold when the variables  $\{x_i\}$  are all Gaussians, in which case  $K_i = 3$  for all  $i$ .

We now compute the PDF for a component of the electric field. We proceed in two steps: we first compute the PDF for a variable of the form

$$s \equiv x_1 x_2, \quad (16)$$

and we then proceed to compute the distribution of

$$z \equiv s_1 - s_2, \quad (17)$$

where  $s_1$  and  $s_2$  are both of the form Eq. (16). We use Eq. (A3) from the Appendix:

$$\begin{aligned} f(s) &= \int f_1(x_1) f_2(x_2) \delta(s - x_1 x_2) dx_1 dx_2 \\ &= \int f_1(x_1) f_2(s/x_1) \frac{dx_1}{|x_1|}, \end{aligned} \quad (18)$$

where  $f(x)$  stands for the PDF of the random variable  $x$ ,  $f_i \equiv f(x_i)$  and we used the relation  $\delta(s - x_1 x_2) = |x_1|^{-1} \delta(x_2 - s/x_1)$ . Note that in Eq. (18) we made use of the statistical independence to write  $f(x_1, x_2) = f_1(x_1) f_2(x_2)$ . We now replace  $f_1$  and  $f_2$  in the integral by Gaussians of dispersions  $\sigma_1$  and  $\sigma_2$ ,

$$f_i \equiv \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{1}{2}\left(\frac{x_i}{\sigma_i}\right)^2\right]. \quad (19)$$

Assuming  $s > 0$ , we change variables  $x_1 \rightarrow u \equiv x_1 / (\sigma_1 \sqrt{2s})$ , so that the integral Eq. (18) reads

$$\begin{aligned} f(s) &= \frac{1}{\pi \sigma_s} \int_0^\infty \exp\left[-s\left(u^2 + \frac{1}{4\sigma_s^2 u^2}\right)\right] \frac{du}{u} \\ &= \frac{1}{\pi \sigma_s} K_0\left(\frac{s}{\sigma_s}\right), \quad s > 0, \end{aligned} \quad (20)$$

where  $\sigma_s \equiv \sigma_1 \sigma_2$  [see Eq. (12)] and  $K_0(x)$  is a modified Bessel function of the second kind. It is evident that  $f(s)$  is an even function of  $s$  [ $f(s) = f(-s)$ ], so we can replace  $s \rightarrow |s|$  to obtain

$$f(s) = \frac{1}{\pi \sigma_s} K_0\left(\frac{|s|}{\sigma_s}\right). \quad (21)$$

We can now compute moments of arbitrary (even) order (odd moments vanish):

$$\langle s^n \rangle = \frac{1}{\pi \sigma_s} \int s^n K_0\left(\frac{|s|}{\sigma_s}\right) ds = \sigma_s^n [(n-1)!!]^2. \quad (22)$$

As a consistency check, we note that the same result can be obtained from Eq. (11) and the well known result for even moments of a Gaussian variable. Also, Eq. (22) implies  $K_s = 9$ , in agreement with Eq. (13).

Now we are in a position of finding the PDF for the components of the induced electric field. We consider a variable of the form [see Eq. (17)]  $z = s_1 - s_2$ , where  $f_i(s_i) = (1/\pi \sigma_{s_i}) K_0(|s_i|/\sigma_{s_i})$ . In this case, Eq. (A3) yields

$$\begin{aligned} f(z) &= \int f_1(s_1) f_2(s_2) \delta(z - s_1 + s_2) ds_1 ds_2 \\ &= \int f_1(s_1) f_2(z - s_1) ds_1, \end{aligned} \quad (23)$$

where we used the parity of  $f_2$ . We immediately notice that the problem reduces to the convolution of two  $K_0$  functions. The convolution theorem for cosine Fourier transforms for even functions  $f_1$  and  $f_2$ , reads

$$\frac{1}{2} \int f_1(s) f_2(z - s) ds = \left(\frac{\pi}{2}\right)^{1/2} F^{-1}[F(f_1)F(f_2)], \quad (24)$$

where  $F$  stands for the cosine Fourier transform. The direct transforms of Eq. (21) are straightforward. Assuming  $z > 0$ ,

$$F(f_i) = \frac{1}{\sqrt{2\pi} \sqrt{1 + (k\sigma_{s_i})^2}}, \quad i = 1, 2. \quad (25)$$

If  $\sigma_{s_1} = \sigma_{s_2}$  [which is ensured by the variance isotropy condition, Eq. (10)], we can compute the inverse transform in Eqs. (23) and (24),

$$f(z) = \frac{1}{2\sigma_{s_1}} \exp\left(-\frac{z}{\sigma_{s_1}}\right), \quad z > 0. \quad (26)$$

But we know that  $f(z) = f(-z)$ , so we can write in general [see Eqs. (8) and (9)]

$$f(c_i) = \frac{1}{\sqrt{2}\sigma_{e_i}} \exp\left(-\frac{\sqrt{2}}{\sigma_{e_i}} |e_i|\right), \quad (27)$$

where, according to Eq. (14)

$$\sigma_{e_i} = \sqrt{2}\sigma_{v_i}\sigma_{v_j}, \quad i \neq j. \quad (28)$$

It can be checked by simple integration from Eq. (27) that  $f(e_i)$  is properly normalized, and that the standard deviation of  $e_i$  is precisely  $\sigma_{e_i}$ .

### B. Dynamo-type correlated Gaussians

There are at least two cases in which the hypothesis of statistical independence of the variables  $x_1, x_2, x_3, x_4$  is inconsistent with the physical situation: (a) in the presence of cross helicity  $H_c \equiv \langle \mathbf{v} \cdot \mathbf{b} \rangle$  and (b) in the presence of an efficient dynamo. This inconsistency is evident in the former case, since statistical independence implies  $H_c = \langle v_i b_i \rangle = \langle v_i \rangle \langle b_i \rangle = 0$  [where we have used Eq. (5) and the standard repeated index summation notation]. In the latter case, the most relevant quantity is the mean induced electric field of the fluctuations, which is precisely  $\langle \mathbf{e} \rangle = -\langle \mathbf{v} \times \mathbf{b} \rangle$ . If the components of  $\mathbf{v}$  and  $\mathbf{b}$  are statistically independent, then  $\langle \mathbf{e} \rangle = 0$ . It is important to note that the converse is not true. For instance,  $\langle e_z \rangle = 0$  implies  $\langle v_x b_y \rangle = \langle v_y b_x \rangle$ , but this does not imply statistical independence.

We study the case  $H_c = 0$ ,  $\epsilon \neq 0$  (dynamo type) in this section, and leave the case  $H_c \neq 0$ ,  $\epsilon = 0$  (cross helicity type) for the following section. The calculations we are going to perform are similar to the ones in the preceding section. We will assume that the pairs  $(x_1, x_2)$  and  $(x_3, x_4)$  are each well described by Gaussian joint PDFs  $f_{12}$  and  $f_{34}$  of the form [13],

$$\begin{aligned} f_{ij} &\equiv \frac{1}{2\pi\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}} \exp\left\{-\frac{1}{2(1-\rho_{ij}^2)}\right. \\ &\quad \left.\times \left[\left(\frac{x_i}{\sigma_i}\right)^2 - 2\rho_{ij}\frac{x_i}{\sigma_i}\frac{x_j}{\sigma_j} + \left(\frac{x_j}{\sigma_j}\right)^2\right]\right\}, \end{aligned} \quad (29)$$

where

$$\rho_{ij} \equiv \frac{\langle x_i x_j \rangle}{\sigma_i \sigma_j}. \quad (30)$$

Note that the mean value of  $z$  can be written in terms of  $\rho_{12}$  and  $\rho_{34}$ ,

$$\langle z \rangle = (\rho_{12} - \rho_{34}) \sigma_1 \sigma_2. \quad (31)$$

That is, the mean induced electric field is nonzero, unless  $\rho_{12} = \rho_{34}$ . We now proceed as in the preceding section. We first compute the PDF for  $s = x_1 x_2$  from Eq. (A3). The calculation is similar to the one leading to Eq. (21), now yielding

$$f(s) = \frac{\sqrt{1-\rho_{12}^2}}{\pi a_{12}} \exp\left(\frac{\rho_{12}}{a_{12}} s\right) K_0\left(\frac{|s|}{a_{12}}\right), \quad a_{12} \equiv (1-\rho_{12}^2)\sigma_1\sigma_2. \quad (32)$$

The next step is to write the PDF for  $z=s_1-s_2$  using Eq. (A3) as in Eq. (23); to make the integral tractable, we assume

$$\rho_{12} = -\rho_{34} \equiv \rho. \quad (33)$$

In this case, the exponentials in front of Eq. (32) cancel their “ $s$ ” dependence when computing the product  $f_1 f_2$  in Eq. (23), contributing just a factor  $\exp[(\rho_{12}/a_{12})z]$ , which drops out of the integral. The problem is thus reduced to the evaluation of the convolution of a function  $K_0(|s|/a_{12})$  with itself, which yields an exponential as we saw in the preceding section. The final result can be expressed as

$$f(e_l) = \frac{1}{2\sigma_{v_i}\sigma_{v_j}} \exp\left[-\frac{|e_l|}{\sigma_{v_i}\sigma_{v_j}(1+\text{sgn}(z)\rho)}\right], \quad (34)$$

where  $\text{sgn}(z)$  is the function “sign of  $z$ ” and  $\rho = \langle e_l \rangle / (2\sigma_{v_i}\sigma_{v_j})$ . Interestingly, the PDF for  $e_l$  is an exponential  $f \sim \exp(-|e_l|/\sigma_+)$  when  $e_l > 0$ , and a different exponential  $f \sim \exp(-|e_l|/\sigma_-)$  when  $e_l < 0$ , with  $\sigma_+/\sigma_- = (1+\rho)/(1-\rho)$ . When  $\rho=0 = \langle e_l \rangle$ , Eq. (34) reduces to the uncorrelated result Eq. (27).

We can compute the standard deviation of  $z=e_l$  from Eq. (34). However, it is better to do it in terms of  $s_1$  and  $s_2$ , which will give us a result for generic  $\rho_1$  and  $\rho_2$ . Equation (30) simply states that  $\langle s_1 \rangle = \rho_{12}\sigma_1^2\sigma_2^2$ , and  $\langle s_1^2 \rangle = (1+2\rho_{12}^2)\sigma_1^2\sigma_2^2$  by integration from Eq. (32). We now use these results (and the analogous forms for  $\langle s_2 \rangle$  and  $\langle s_2^2 \rangle$ ) to obtain

$$\sigma_{e_l} = \sigma_z = \sigma_1\sigma_2\sqrt{2+\rho_{12}^2+\rho_{34}^2}. \quad (35)$$

Note that Eq. (35) reduces to Eq. (14) when  $\rho_{12}=0 = \rho_{34}$ .

We note that a realistic dynamo may depart significantly from our calculations. Recent results in dynamo theory in fact suggest that Alfvén waves play a key role in magnetic field generation [14,15]. This in turn implies a strong local correlation (neglected in this section) between the velocity and magnetic fields, since Alfvén waves satisfy  $\mathbf{u} \cdot \mathbf{b} / |\mathbf{u}||\mathbf{b}| = \pm 1$ .

### C. Cross helicity-type correlated Gaussians

We focus now on a model for the case in which the cross helicity is finite (see the discussion in the preceding section) and the mean induced electric field is zero (or  $H_c \neq 0$ ,  $\epsilon = 0$ ). The cross helicity is one of the rugged ideal invariants in MHD and is present in many real world turbulent systems. The inner solar wind, for example, has non-negligible cross helicity [16] as a general rule, although in some cases [17,18], especially in the outer heliosphere [19], the cross helicity is very small. In the extreme case in which the cross helicity is maximum (i.e., when the normalized cross helicity is  $\pm 1$ ), however, no turbulence is possible, since the nonlinear terms in the MHD equations completely vanish [20].

In terms of the  $\{x_i\}$  variables, we will have to consider correlations for the pairs  $(x_1, x_4)$  and  $(x_2, x_3)$ . We apply Eq. (A3) in just one step,

$$f(z) = \int f_{14}(x_1, x_4) f_{23}(x_2, x_3) \times \delta[z - (x_1 x_2 - x_3 x_4)] dx_1 dx_2 dx_3 dx_4, \quad (36)$$

where both  $f_{14}$  and  $f_{23}$  are given by Eq. (29). It is trivial to verify that in this case there is no mean electric field (for instance,  $\langle z \rangle = \langle x_1 x_2 \rangle - \langle x_3 x_4 \rangle = \langle x_1 \rangle \langle x_2 \rangle - \langle x_3 \rangle \langle x_4 \rangle = 0$ ). The standard deviation for the components of the electric field can be computed very easily as well,

$$\sigma_z = \sigma_1\sigma_2\sqrt{2(1-\rho_{14}\rho_{23})}. \quad (37)$$

After integration in Eq. (36) we obtain (see the Appendix for details)

$$f(z) = \frac{1}{\sqrt{2}\sigma_z} g \exp\left(\rho_{14}, \rho_{23}, -\frac{\sqrt{2}}{\sigma_z}|z|\right), \quad (38)$$

where  $\sigma_z$  is given by Eq. (37) and we defined

$$g \exp(\rho_1, \rho_2, z) \equiv \frac{\sqrt{1-\rho_1\rho_2}}{2\pi} \int_0^{2\pi} \exp\left(\frac{\Theta_1}{\Theta_2} \sqrt{[1-\rho_1\rho_2]/[1-\rho_1^2]} z\right) \times \frac{d\theta}{\Theta_1\Theta_2}, \quad (39)$$

$$\Theta_i \equiv \sqrt{1-\rho_i \sin(2\theta)}. \quad (40)$$

Note that when  $\rho_1 = \rho_2$ , the exponential  $\exp(z)$  drops out of the integral in Eq. (39), giving the result

$$g \exp(\rho, \rho, z) \equiv e^z. \quad (41)$$

It is evident from Eqs. (29) and (36) that

$$g \exp(\rho_1, \rho_2, z) = g \exp(\rho_2, \rho_1, z). \quad (42)$$

Figure 1 shows  $g \exp(\rho_1, \rho_2, -|z|)$ , evaluated numerically for the cases  $\rho_1 = \rho_2 = 0.9$  and  $\rho_1 = -\rho_2 = 0.9$ . The first case corresponds to an exponential, as indicated by Eq. (41), and hence has a kurtosis of 6. The other case shows a flatter function, yielding a kurtosis of 8.89. In fact, when  $\rho_1 \rightarrow 1$  and  $\rho_2 \rightarrow -1$ , and  $z \rightarrow 2x_1 x_2$ , which implies, according to Eq. (21), that  $f(z) \rightarrow (\pi\sigma_z)^{-1} K_0(|z|/\sigma_z)$ . That is,

$$\lim_{\rho \rightarrow 1} g \exp(\rho, -\rho, -|z|) = \frac{\sqrt{2}}{\pi} K_0(|z|). \quad (43)$$

In particular, in this limit the kurtosis is 9 [see Eq. (13)].

### D. Variance anisotropic turbulence

So far we have assumed the validity of Eq. (10), that is,  $\sigma_{v_i}\sigma_{b_j} = \sigma_{v_j}\sigma_{b_i}$ . We note that even though variance isotropy implies Eq. (10), such a relationship does not require vari-

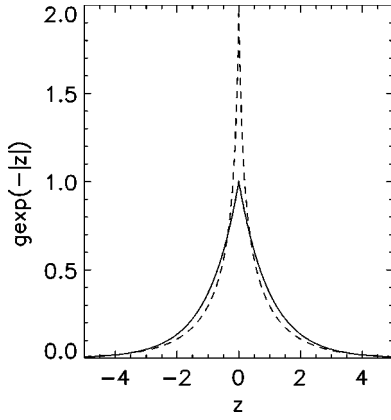


FIG. 1. Function  $g \exp(\rho_1, \rho_2, -|z|)$  for different parameters:  $(\rho_1, \rho_2) = (0.9, 0.9)$  in solid line, and  $(0.9, -0.9)$  in dashed. The kurtosis for these two cases is, respectively, 6 and 8.8.

ance isotropy. For instance, consider the situation of compressible MHD turbulence in presence of a dc field in the  $z$  direction. In this case,  $\sigma_{v_x} \approx \sigma_{v_y} \gg \sigma_{v_z}$ , and  $\sigma_{b_x} \approx \sigma_{b_y} \gg \sigma_{b_z}$  [12]. We immediately see that the variance isotropic results are valid for  $e_z$  due to variance isotropy in the perpendicular planes (variance axisymmetry). For the perpendicular components of  $\mathbf{e}$ , on the other hand, the variance isotropic results are valid if  $\sigma_{v_x}/\sigma_{v_y} = \sigma_{b_z}/\sigma_{b_x}$  and  $\sigma_{v_z}/\sigma_{v_x} = \sigma_{b_y}/\sigma_{b_z}$ , which might be approximately satisfied in some situations.

Even though we do not find analytical results for the general anisotropic case, there is a limit that is analytically tractable. This is the limit in which

$$\lambda \equiv \frac{\sigma_3 \sigma_4}{\sigma_1 \sigma_2} \rightarrow 0. \quad (44)$$

Note that this limit would occur, for instance, if one of the fluctuating fields was plane polarized, but the other was variance isotropic, in this case, the components of  $\mathbf{e}$  in the polarization plane would both satisfy Eq. (44). Let us write Eq. (A3) in the form

$$f(z) = \int F\left(\frac{x_1}{\sigma_1}, \frac{x_2}{\sigma_2}, \frac{x_3}{\sigma_3}, \frac{x_4}{\sigma_4}\right) \times \delta[z - (x_1 x_2 - x_3 x_4)] dx_1 dx_2 dx_3 dx_4, \quad (45)$$

where  $F(x_1, x_2, x_3, x_4)$  is the joint PDF for  $x_1, x_2, x_3, x_4$ . Making the change of variables  $\tilde{x}_i \equiv x_i/\sigma_i$ ,  $\tilde{z} \equiv z/(\sigma_1 \sigma_2)$ , we can rewrite Eq. (45) as

$$f(\tilde{z}) = \sigma_3 \sigma_4 \int F\left(\frac{\tilde{z} - \lambda \tilde{x}_3 \tilde{x}_4}{\tilde{x}_2}, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right) \frac{d\tilde{x}_2}{|\tilde{x}_2|} d\tilde{x}_3 d\tilde{x}_4, \quad (46)$$

and taking the limit  $\lambda \rightarrow 0$ ,

$$f(\tilde{z}) \rightarrow \sigma_3 \sigma_4 \int F\left(\frac{\tilde{z}}{\tilde{x}_2}, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right) \frac{d\tilde{x}_2}{|\tilde{x}_2|} d\tilde{x}_3 d\tilde{x}_4. \quad (47)$$

Note that the dependence in  $(\tilde{x}_3, \tilde{x}_4)$  can be integrated in a first step, and finally the problem reduces to an integral in

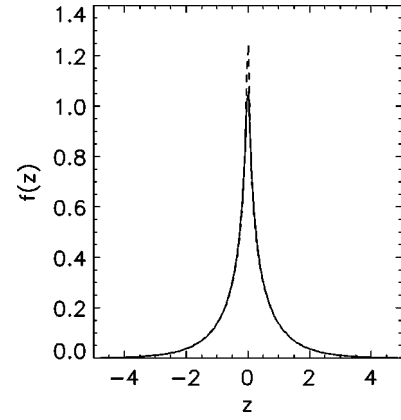


FIG. 2. In solid line, the PDF for a random variable  $z \equiv x_1 x_2 - x_3 x_4$ . The  $\{x_i\}$  variables are generated with a Gaussian random number generator, and satisfy  $\lambda \equiv \sigma_3 \sigma_4 / \sigma_1 \sigma_2 = 0.1$  [see Eq. (44)]. In dashed line: the exact result for the PDF of  $z$  in the limit  $\lambda \rightarrow 0$  [see Eq. (48)].

$\tilde{x}_2$ . It is straightforward to show that both for uncorrelated Gaussians and cross-helicity-type correlated Gaussians we obtain [see Eq. (21)]

$$f(e_l) \rightarrow \frac{1}{\pi \sigma_{v_i} \sigma_{v_j}} K_0\left(\frac{|e_l|}{\sigma_{v_i} \sigma_{v_j}}\right) \quad (48)$$

and for dynamo-type correlated Gaussians we obtain [see Eq. (32)]

$$f(e_l) \rightarrow \frac{\sqrt{1-\rho^2}}{\pi a} \exp\left(\frac{\rho}{a} z\right) K_0\left(\frac{|z|}{a}\right), \quad a \equiv (1-\rho^2) \sigma_{v_i} \sigma_{v_j} \quad (49)$$

and  $\rho = \langle e_l \rangle / \sigma_{v_i} \sigma_{v_j}$ . Note that all the solutions in this limit are related to  $K_0$  functions. We recall that  $K_0$  diverges logarithmically at the origin, and decays faster than a simple exponential,

$$K_0(|z|) \approx -\ln\left(\frac{|z|}{2}\right) + C, \quad |z| \ll 1, \quad (50)$$

$$K_0(z) \approx \left(\frac{\pi}{2}\right)^{1/2} \frac{e^{-|z|}}{\sqrt{|z|}}, \quad |z| \gg 1, \quad (51)$$

$C \approx -0.577215$  is Euler's constant.

Figure 2 shows the distribution function  $f(z)$  for small but finite  $\lambda = 0.1$ . The Gaussian random variables  $\{x_i\}$  for the plot are obtained with a Gaussian random number generator. Except for the discrepancy at the origin, the PDF for  $z$  is fairly close to the asymptotic solution Eq. (48). The plot shows that  $\lambda$  does not need to get too small for the PDF to be close to Eq. (48).

### III. NUMERICAL RESULTS

We solve the standard MHD incompressible dissipative equations using a pseudospectral Fourier technique as described in Ref. [21]. We label our runs as follows. (i) ISO

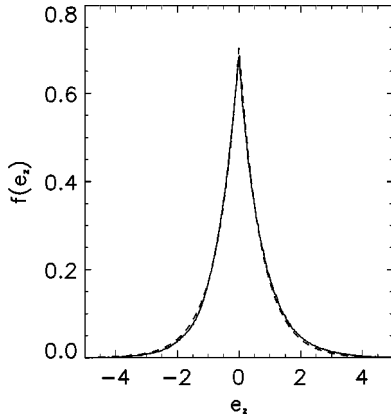


FIG. 3. In solid line, the PDF for one component of the electric field (in units of its standard deviation), as computed from the isotropic simulation at  $t=3t_0$ . In dashed line: the exponential PDF predicted by Eq. (27).

run. This is an isotropic run, with no dc field. We let  $\mathbf{v}(\mathbf{x},t)$  and  $\mathbf{b}(\mathbf{x},t)$  relax in time, from an isotropic initial condition with no net mean magnetic field.

(ii) dc run. We let a dc magnetic field  $\mathbf{B}_0=B_0\hat{\mathbf{z}}$  act on the plasma, and study the relaxation of the MHD field  $\mathbf{v}(\mathbf{x},t)$  and  $\mathbf{B}(\mathbf{x},t)=\mathbf{B}_0+\mathbf{b}(\mathbf{x},t)$ .

Both runs start with isotropic, power-law, random-phase Gaussian ( $\mathbf{v},\mathbf{b}$ ) fields, with unit energy, zero cross helicity and zero magnetic helicity. Note that these simulations should not be expected to produce significant dynamo activity since there is a lack of helicity [1,22] in the preparation of the system and there is no forcing. More explicitly, the initial fields are such that

$$E^v(k)=E^b(k)=\frac{C}{1+(k/k_{\text{knee}})^{5/3}}, \quad (52)$$

where  $k_{\text{knee}}=4k_0$ ,  $k_0$  being the smallest wave number of the system.  $E^v(k)$  and  $E^b(k)$  are, respectively, the kinetic and the magnetic omnidirectional power spectra. We define  $L_0=2\pi/k_{\text{knee}}$ , our “energy containing scale,” as the unit length,  $u_0$  as the unit velocity and magnetic field (recall that the magnetic field is written in velocity units) and  $l_0=L_0/u_0$  as the unit time. The normalization constant  $C$  in Eq. (52) is chosen so that  $\langle b^2 \rangle = \langle v^2 \rangle = 2\sum_k E^{v,b}(k) = 1$  (at  $t=0$ ). For the dc run,  $B_{\text{dc}}=1$  as well. The magnetic Prandtl number is set to 1, the macroscopic Reynolds number is  $R=u_0L_0/\nu=200$  and the resolution is  $128^3$  in all the runs. In all cases the solutions approximately satisfy variance isotropy:  $\langle v_x^2 \rangle \approx \langle v_y^2 \rangle \approx \langle v_z^2 \rangle$ ,  $\langle b_x^2 \rangle \approx \langle b_y^2 \rangle \approx \langle b_z^2 \rangle$ , where the brackets  $\langle \rangle$  mean here spatial average over the whole computational volume. The fluctuating fields  $\mathbf{v}$  and  $\mathbf{b}$  average to zero at all times:  $\langle \mathbf{v} \rangle = 0 = \langle \mathbf{b} \rangle$ . These simulations were recently analyzed in the context of local development of anisotropy in MHD [23].

Figure 3 shows the PDF for  $e_z$  for the ISO run, at  $t=3$ . The PDF is remarkably close to the exponential function Eq. (27), as it turns out from the figure. The dc run yields a similar result, as shown in Fig. 4. The PDFs for the compo-

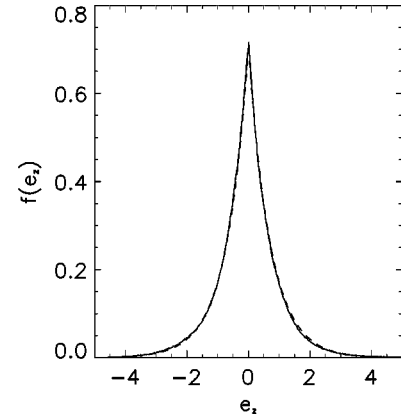


FIG. 4. In solid line, the PDF for one component of the electric field (in units of its standard deviation), as computed from the dc simulation at  $t=3t_0$ . In dashed line: the exponential PDF predicted by Eq. (27).

nents of  $\mathbf{v}$  and  $\mathbf{b}$  (not shown) always lie close to a Gaussian, which is consistent with the quasinormal assumption of this work, and in general with the quasinormal character of the turbulence.

A way to quantify the departure of the observed PDFs from the predicted values is to measure their kurtosis. The kurtosis for the components of  $\mathbf{e}$  has an interesting behavior, shown in Figs. 4 and 5. It seems evident that the kurtosis tends to be smaller than 6 in the ISO case, and greater than 6 in the dc case. Nonetheless, the departure from the predicted value of 6 is always relatively small. Moreover, the departure of the fields  $\mathbf{v}$  and  $\mathbf{b}$  themselves from Gaussianity is very small, with typical values for the kurtosis of their components between 3 and 3.1. We estimate the intrinsic statistical error of these measurements as the standard deviation of the set of values of the kurtosis for the components of  $\mathbf{v}$  and  $\mathbf{b}$  from the initial condition (which we recall are generated numerically as uncorrelated Gaussian variables). We thus obtain a value of 0.05 for the statistical error in computing  $K(v_i)$  and  $K(b_i)$ . We finally note that for  $t>0$ , about two-

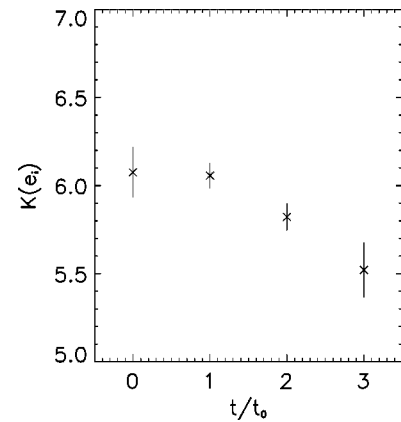


FIG. 5. In solid line, the PDF for one component of the electric field (in units of its standard deviation), as computed from the isotropic simulation. For each value of time, the values  $K(e_x)$ ,  $K(e_y)$ , and  $K(e_z)$  are used to compute a mean value (displayed as a cross) and the statistical error (displayed as a vertical bar).

thirds of the values of the kurtosis for the components of  $\mathbf{v}$  and  $\mathbf{b}$  in our simulations are outside the range  $3.00 \pm 0.05$ . We conclude that the evolution of our fields is very close to, but not trivially Gaussian.

#### IV. SUMMARY AND CONCLUSIONS

In the present paper we study the statistical properties of the induced turbulent electric field for a magnetohydrodynamic (MHD) plasma under the quasynormal approximation. In Sec. II we assume exact Gaussianity for both the turbulent velocity field  $\mathbf{v}$  and the turbulent magnetic field  $\mathbf{b}$ , and find analytical expressions for the PDF for the components of the fluctuating induced electrical field  $\mathbf{e} = -\mathbf{v} \times \mathbf{b}$  in some cases of interest. The simplest and most appealing result is obtained when statistical independence is assumed for the components of  $\mathbf{v}$  and  $\mathbf{b}$ . In this case, the PDF for  $e_i$  is simply an exponential, see Eq. (27). As mentioned above, statistical independence implies zero cross helicity  $H_c$ , and zero mean turbulent induced electric field  $\langle \mathbf{e} \rangle$  (i.e., impossibility of a dynamo effect). In Secs. II B and II C we allow correlations between the components of  $\mathbf{v}$  and  $\mathbf{b}$ , to extend the model, respectively, to the cases of finite  $\langle \mathbf{e} \rangle$  and finite  $H_c$ . In the former case, we obtain an analytical result when  $\langle v_i b_j \rangle = -\langle v_j b_i \rangle$ , i.e., Eq. (34): the PDF turns out to be an exponential of the form  $\exp(-|e_i|/\sigma_+)$  for  $e_i > 0$ , and an exponential of the form  $\exp(-|e_i|/\sigma_-)$  for  $e_i < 0$ . The PDF for the latter case can be written in terms of a function  $g \exp(\rho_1, \rho_2, z)$ , which also reduces to an exponential when PDF when  $\langle v_i b_j \rangle = \langle v_j b_i \rangle$ , as indicated by Eqs. (38)–(41).

All of these results assume variance isotropy for the turbulent fluctuations. In Sec. II D we show that the effects of variance anisotropy are measured by a single parameter  $\lambda = \sigma_{v_j} \sigma_{b_i} / \sigma_{v_i} \sigma_{b_j}$ . We find that for extreme variance anisotropy ( $\lambda \rightarrow 0$ ), the three cases considered before (i.e., uncorrelated Gaussians, dynamotype correlated Gaussians and cross-helicity-type Gaussians) give solutions related to modified Bessel functions  $K_0$  [see Eqs. (48) and (49)].

The use of exact normality in our calculations raises several warnings. To begin with, the PDF allows computation of statistical moments of arbitrary order, while it is very well known that higher-order moments are usually more prone to depart from Gaussian statistics than lower-order moments are. Another point to consider is the fact that statistics for two-point correlations are usually very far from Gaussian if the separation between the correlated points is in (or close to) the dissipation range (see for instance [6,11]). In this regard our one-point calculations are in the most favorable case, i.e., the limit in which the separation tends to the integral scale and the statistics are closer to being Gaussian. But any extension of our calculations to two-point correlations needs to be done carefully. Finally, it needs to be kept in mind that closure theories (more specifically the EDQNM) had to refine the quasynormal approximation to the point of adding a phenomenological eddy viscosity and Markovianization in order to obtain physical results for the evolution of the energy spectrum. Even though this issue is also related to two-point statistics, it should not be disregarded.

In Sec. III we report results from three-dimensional nu-

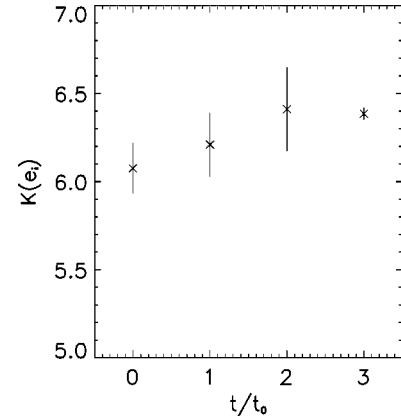


FIG. 6. In solid line, the PDF for one component of the electric field (in units of its standard deviation), as computed from the dc simulation. For each value of time, the values  $K(e_x)$ ,  $K(e_y)$ , and  $K(e_z)$  are used to compute a mean value (displayed as a cross) and the statistical error (displayed as a vertical bar).

merical simulations of the MHD equations at moderate Reynolds numbers, where we investigate turbulent relaxation from a broad-band initial condition, with no cross helicity. The PDFs for the components of  $\mathbf{e}$  are remarkably close to the exponential function predicted in Eq. (27), as shown in Figs. 3 and 4. The dynamical departure from the exponential PDF seems to depend on the presence of a background, dc magnetic field. Figures 5 and 6 seem to indicate that in the absence of a dc field the kurtosis is slightly smaller than 6, while in the presence of a dc field it is slightly greater than 6. More numerical exploration may be needed to confirm this trend. But in general, the comparison between the numerical and the analytical results allows us to be confident that the analytical results give a very good first-order approximation to the problem, at least at moderate Reynolds numbers.

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#### APPENDIX: SOME USEFUL RELATIONS

Let  $z$  be a random variable that depends on another random variable  $x$ , then

$$f(z) = \int p(z|x)f(x)dx, \quad (\text{A1})$$

where  $f$  is a PDF for its variable and  $p(z|x)$  is the conditional probability distribution for  $z$  given  $x$ . Consider now the special case in which  $z$  is deterministically related to  $x$  by a known function  $z = \psi(x)$ . In this case,  $p(z|x) = \delta(z - \psi(x))$ ,

$$f(z) = \int f(x) \delta(z - \psi(x)) dx. \quad (\text{A2})$$

The extension of this result to multiple dimensions is straightforward. If  $\{x_i\}$  is a set of  $n$  random variables  $\{x_1 \cdots x_n\}$ , then

$$f(z) = \int f(x_1, \dots, x_n) \delta(z - \psi(x_1, \dots, x_n)) dx_1 \cdots dx_n, \quad (\text{A3})$$

where  $f(x_1, \dots, x_n)$  is the joint PDF of the  $\{x_i\}$  variables.

We now show how to obtain the PDF for the cross-helicity-type case, Eq. (38), from Eq. (36). Our first step is to make a change to nondimensional variables, as in Eq. (46). We note that here  $\lambda = 1$  (i.e., we are assuming variance isotropy). We replace  $\delta(z - (x_1 x_2 - x_3 x_4)) = \delta(\sigma_1 \sigma_2 [\tilde{z} - (\tilde{x}_1 \tilde{x}_2 - \tilde{x}_3 \tilde{x}_4)]) = (1/|\tilde{x}_1| \sigma_1 \sigma_2) \delta(\tilde{x}_2 - \tilde{z}/\tilde{x}_1 - \tilde{x}_3 \tilde{x}_4 / \tilde{x}_1)$  in Eq. (36) and compute the trivial integral in  $\tilde{x}_2$ . We then compute the

integral on  $\tilde{x}_3$ , which is of the form  $\int \exp(-A\tilde{x}_3 - B\tilde{x}_3^2) d\tilde{x}_3 = \sqrt{\pi/B} \exp(A^2/4B)$ , where both  $A$  and  $B > 0$  are constant with respect to  $\tilde{x}_3$ . Rearranging terms and making a change to polar variables  $(\tilde{x}_1, \tilde{x}_4) \rightarrow (r, \theta)$ , we can write the remaining two-dimensional integral as

$$f(\tilde{z}) = \frac{1}{(2\pi)^{3/2} \sigma_1 \sigma_2 \sqrt{1 - \rho_{14}^2}} \int \exp \left[ -\frac{\Theta_{14}^2}{2(1 - \rho_{14}^2)} r^2 - \frac{\tilde{z}^2}{2\Theta_{23}^2} r^{-2} \right] \frac{dr d\theta}{\Theta_{23}}, \quad (\text{A4})$$

where we defined  $\Theta_{ij} \equiv \sqrt{1 - \rho_{ij} \sin(2\theta)}$ . Making a last change in variables  $r \rightarrow \hat{r}$  according to  $[\Theta_{14}^2/2(1 - \rho_{14}^2)] r^2 = |\tilde{z}| \hat{r}^2$ , the integral in  $\hat{r}$  takes the form  $\int \exp(-|\tilde{z}|(\hat{r}^2 - A^2/4\hat{r}^2)) d\hat{r} \propto \exp(A|\tilde{z}|)$ , where  $A$  depends on  $\theta$ . The exact result is shown in Eq. (38).

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